# Detailed formulation of SCALE-DG

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# 1 Introduction

## 1.1 What is SCALE-DG?

FE-project provides a library and sample programs for the discontinuous Galerkin methods (DGM). Furthermore, we provide an atmospheric model with a regional/global dynamical core based on DGM, SCALE-DG. In FE-Project, we use scalable Computing for Advanced Library and Environment (SCALE), which is a basic software library of weather and climate models of the earth and planets intended for widespread use.

The general references of SCALE-DG are Kawai and Tomita (2023, 2025). If SCALE-DG is used in your studies, please cite two our papers in addition to the reference papers of SCALE library (Nishizawa et al., 2015; Sato et al., 2015).

# 2 Governing equations

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## 2.1 Coordinate system

To describe the governing equations for both regional and global dynamical cores, a non-orthogonal curvilinear horizontal coordinate $(\xi, \eta)$ and a general vertical coordinate $\zeta$ are introduced, following Li et al. (2020). For the horizontal coordinate transformation, the contravariant form of the metric tensor is represented by $G_h^{ij}$ for $i, j = 1, 2$. A three-dimensional metric tensor with the horizontal coordinate transformation is defined as

$$G^{ij} = \begin{pmatrix} G_h^{11} & G_h^{12} & 0 \\ G_h^{21} & G_h^{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.1}$$

The horizontal Jacobian is defined as $\sqrt{G_h} = |G_h^{ij}|^{-\frac{1}{2}}$. The Christoffel symbol of the second kind $\Gamma^i_{ml}$, which means the spatial variation of the basis vector, is represented as

$$\Gamma^i_{ml} = \frac{1}{2} G^{im} \left( \frac{\partial G_{jm}}{\partial x^k} + \frac{\partial G_{km}}{\partial x^j} + \frac{\partial G_{jk}}{\partial x^m} \right) \tag{2.2}$$

For the vertical coordinate transformation, the metric tensor is defined as $G_v^{13} = \partial\zeta/\partial\xi, G_v^{23} = \partial\zeta/\partial\eta$ and the vertical Jacobian is defined as $\sqrt{G_v} = \partial z/\partial\zeta$. The vertical velocity in the transformed vertical coordinate can be written using contravariant components of wind vector $(u^\xi, u^\eta, u^\zeta)$ as

$$\widetilde{u^\zeta} \equiv \frac{d\zeta}{dt} = \frac{1}{\sqrt{G_v}} \left( u^\zeta + \sqrt{G_v} G_v^{13} u^\xi + \sqrt{G_v} G_v^{23} u^\eta \right). \tag{2.3}$$

The final Jacobian composed of horizontal and vertical coordinate transformations can be represented as $\sqrt{G} = \sqrt{G_h}\sqrt{G_v}$. Hereafter, to briefly describe the formulations, the coordinate variables are sometimes expressed using $(\xi^1, \xi^2, \xi^3) = (\xi, \eta, \zeta)$. In addition, the Einstein summation notation will be applied for repeated indices when representing the geometric relations.

### 2.1.1 Horizontal coordinates used in regional model

For horizontal coordinates in our regional model, the horizontal Cartesian coordinates $(x, y)$ is simply adopted although we will introduce map projections in the near future.

Then,

$$G_h^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sqrt{G_h} = 1, \tag{2.4}$$

The Christoffel symbol of the second kind $\Gamma^i_{ml}$ is represented as

$$\Gamma^1_{ml} = 0, \ \ \Gamma^2_{ml} = 0, \ \ \Gamma^3_{ml} = 0, \tag{2.5}$$

where $m, l = 1, 2, 3$.

The components of angular velocity vector included in the Coriolis terms are given as

$$\begin{aligned} \Omega^1 = 0, \ \ \Omega^2 = 0, \\ \Omega^3 = f_0 + \beta y. \end{aligned} \tag{2.6}$$

Here, $f_0 = 2\Omega \sin\theta_0, \beta = 2\Omega \cos\theta_0$ where $\omega$ is the angular velocity of the planet and $\theta_0$ is the reference latitude.

### 2.1.2 Horizontal coordinates used in global model

For horizontal coordinates in our global model, we adopt an equiangular gnomonic cubed-sphere projection (Sadourny, 1972; Ronchi et al., 1996) to map a cube onto a sphere. Compared to a conformal projection (Rančić et al., 1996), we prefer this projection to generate more uniform grids in high spatial resolutions, although the non-orthogonal basis need to be treated. In each panel of the cube, a local coordinate using the central angles $(\alpha, \beta)$ $(\in [-\pi/4, \pi/4])$ was introduced and related to the horizontal coordinates $(\xi, \eta)$ by $\xi = \alpha, \eta = \beta$. Based on the derivation with the coordinate transformation in previous studies (e.g., Nair et al., 2005; Ullrich and Jablonowski, 2012b; Li et al., 2020), the horizontal contravariant metric tensor and the horizontal Jacobian for the equiangular gnomonic cubed-sphere projection can be written as, respectively,

$$G_c^{ij} = \frac{\delta^2}{r^2(1+X^2)(1+Y^2)} \begin{pmatrix} 1+Y^2 & XY \\ XY & 1+X^2 \end{pmatrix}, \quad \sqrt{G_c} = \frac{r^2(1+X^2)(1+Y^2)}{\delta^3}, \tag{2.7}$$

where $X = \tan\alpha$, $Y = \tan\beta$, $\delta = \sqrt{1+X^2+Y^2}$, and $r$ is the radial coordinate. The Christoffel symbol of the second kind $\Gamma^i_{ml}$ is represented as

$$\begin{aligned}
\Gamma^1_{ml} &= \begin{pmatrix} \dfrac{2XY^2}{\delta^2} & \dfrac{-Y(1+Y^2)}{\delta^2} & \dfrac{\delta_S}{r} \\ \dfrac{-Y(1+Y^2)}{\delta^2} & 0 & 0 \\ \dfrac{\delta_S}{r} & 0 & 0 \end{pmatrix}, \\
\Gamma^2_{ml} &= \begin{pmatrix} 0 & \dfrac{-X(1+X^2)}{\delta^2} & 0 \\ \dfrac{-X(1+X^2)}{\delta^2} & \dfrac{2X^2Y}{\delta^2} & \dfrac{\delta_S}{r} \\ 0 & \dfrac{\delta_S}{r} & 0 \end{pmatrix}, \\
\Gamma^3_{ml} &= \delta_S \frac{r(1+X^2)(1+Y^2)}{\delta^4} \begin{pmatrix} -(1+X^2) & XY & 0 \\ XY & -(1+Y^2) & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned} \tag{2.8}$$

where $\delta_S$ is a switch for shallow atmosphere approximation.

The components of angular velocity vector included in the Coriolis terms are given as

$$\Omega^1 = 0, \quad \Omega^2 = \delta_S \frac{\omega\delta}{r(1+Y^2)}, \quad \Omega^3 = \omega\frac{Y}{\delta}, \qquad \text{for the equatorial panels,}$$
$$\Omega^1 = -\delta_S \frac{s\omega X\delta}{r(1+X^2)}, \quad \Omega^2 = -\delta_S \frac{s\omega Y\delta}{r(1+Y^2)}, \quad \Omega^3 = \frac{s\omega}{\delta}, \qquad \text{for the polar panels,} \tag{2.9}$$

where $\omega$ is the angular velocity of the planet and an index $s$ has a value of 1 and -1 for the Northern and Southern polar panels, respectively.

### 2.1.3 Vertical coordinates

To treat the topography, we adopt the traditional terrain-following coordinate (Phillips, 1957; Gal-Chen and Somerville, 1975) as a general vertical coordinate. The vertical coordinate transformation can be expressed as

$$\zeta = z_T \frac{z-h}{z_T - h}, \tag{2.10}$$

where $z$ is the height coordinate, $z_T$ is the top height of computational domain (we assume it is a constant value), and $h$ is the surface height. The corresponding Jacobian and metric tensor can be represented as

$$\sqrt{G_v} = 1 - \frac{h}{z_T}, \quad \sqrt{G_v}G_v^{13} = \left(\frac{\zeta}{z_T} - 1\right)\frac{\partial h}{\partial \xi}, \quad \sqrt{G_v}G_v^{23} = \left(\frac{\zeta}{z_T} - 1\right)\frac{\partial h}{\partial \eta}, \tag{2.11}$$

respectively.

## 2.2 Governing equations for atmospheric dynamical core

As the governing equations, we adopt the three-dimensional, fully compressible nonhydrostatic equations based on the flux form (e.g., Ullrich and Jablonowski, 2012b). The compact form of the governing equations can be written as

$$\frac{\partial \boldsymbol{q}}{\partial t} + \frac{\partial\left[\boldsymbol{f}(\boldsymbol{q}) + \boldsymbol{f}_{\mathrm{SGS}}(\boldsymbol{q}, \nabla\boldsymbol{q})\right]}{\partial \xi} + \frac{\partial\left[\boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{g}_{\mathrm{SGS}}(\boldsymbol{q}, \nabla\boldsymbol{q})\right]}{\partial \eta} + \frac{\partial\left[\boldsymbol{h}(\boldsymbol{q}) + \boldsymbol{h}_{\mathrm{SGS}}(\boldsymbol{q}, \nabla\boldsymbol{q})\right]}{\partial \zeta}$$
$$= \boldsymbol{S}(\boldsymbol{q}) + \boldsymbol{S}_{\mathrm{SGS}}(\boldsymbol{q}, \nabla\boldsymbol{q}), \tag{2.12}$$

Here, $\boldsymbol{q}$ is the solution vector defined as

$$\boldsymbol{q} = \left(\sqrt{G}\rho', \sqrt{G}\rho u^\xi, \sqrt{G}\rho u^\eta, \sqrt{G}\rho u^\zeta, \sqrt{G}(\rho\theta)', \rho q_*\right)^T, \tag{2.13}$$

where $\rho, \theta$ denote the density and potential temperature defined later, respectively. $q_*$ represents the mass concentration of each material such as water vapor ($q_v$), liquid water ($q_l$), and solid water ($q_s$). The mass concentrations should be the relation as

$$q_d + \sum_{*=v,l,s} q_* = 1, \tag{2.14}$$

where $q_d$ is the mass concentration of dry air. To accurately treat nearly balanced flows, the density $\rho$ and pressure $p$ (thus $\rho\theta$) are decomposed as $q(\xi,\eta,\zeta,t) = q_{\rm hyd}(\xi,\eta,\zeta)+q'(\xi,\eta,\zeta,t)$, where $q_{\rm hyd}$ denotes a variable satisfying the hydrostatic balance and $q'$ denotes the deviation.

In Eq. (2.12), $\boldsymbol{f}(\boldsymbol{q})$, $\boldsymbol{g}(\boldsymbol{q})$, and $\boldsymbol{h}(\boldsymbol{q})$ are inviscid fluxes in the $\xi$, $\eta$, and $\zeta$ directions, respectively. The horizontal inviscid fluxes are represented as

$$\boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} \sqrt{G}\rho u^\xi \\ \sqrt{G}(\rho u^\xi u^\xi + G_h^{11} p') \\ \sqrt{G}(\rho u^\eta u^\xi + G_h^{21} p') \\ \sqrt{G}\rho u^\zeta u^\xi \\ \sqrt{G}\rho\theta u^\xi \\ \sqrt{G}\rho q_* u^\xi \end{pmatrix}, \quad \boldsymbol{g}(\boldsymbol{q}) = \begin{pmatrix} \sqrt{G}\rho u^\eta \\ \sqrt{G}(\rho u^\xi u^\eta + G_h^{12} p') \\ \sqrt{G}(\rho u^\eta u^\eta + G_h^{22} p') \\ \sqrt{G}\rho u^\zeta u^\eta \\ \sqrt{G}\rho\theta u^\eta \\ \sqrt{G}\rho q_* u^\eta \end{pmatrix}, \tag{2.15}$$

and the vertical inviscid fluxes are represented as

$$\boldsymbol{h}(\boldsymbol{q}) = \begin{pmatrix} \sqrt{G}\rho \widetilde{u^\zeta} \\ \sqrt{G}[\rho u^\xi \widetilde{u^\zeta} + (G_v^{13} G_h^{11} + G_v^{23} G_h^{12}) p'] \\ \sqrt{G}[\rho u^\eta \widetilde{u^\zeta} + (G_v^{13} G_h^{21} + G_v^{23} G_h^{22}) p'] \\ \sqrt{G}\rho u^\zeta \widetilde{u^\zeta} + \sqrt{G}_h p' \\ \sqrt{G}\rho\theta \widetilde{u^\zeta} \\ \sqrt{G}\rho q_* \widetilde{u^\zeta} \end{pmatrix}. \tag{2.16}$$

Furthermore, $\boldsymbol{S}(\boldsymbol{q})$ in Eq. (2.12) represents the source terms as

$$\boldsymbol{S}(\boldsymbol{q}) = \begin{pmatrix} \sqrt{G} S_{\rho,\rm phy} \\ \sqrt{G}(F_H^1 + F_M^1 + F_C^1) + \sqrt{G} S_{\rho u^\xi,\rm phy} \\ \sqrt{G}(F_H^2 + F_M^2 + F_C^2) + \sqrt{G} S_{\rho u^\eta\,\rm phy} \\ \sqrt{G}(F_{\rm buo} + F_C^3) + \sqrt{G} S_{\rho u^\zeta,\rm phy} \\ \sqrt{G} S_{\rho\theta,\rm phy} \\ \sqrt{G} S_{\rho q_*,\rm phy} \end{pmatrix}, \tag{2.17}$$

where $F_H^i$ for $i = 1, 2$ are the horizontal pressure gradient terms with hydrostatic balance and can be expressed as

$$F_H^i = -\frac{G_h^{im'}}{\sqrt{G_v}} \left[ \frac{\partial(\sqrt{G_v} p_{\rm hyd})}{\partial \xi^{m'}} + \frac{\partial(G_v^{m'3} \sqrt{G_v} p_{\rm hyd})}{\partial \xi^3} \right]. \tag{2.18}$$

Here, note that $m' = 1, 2$. $F_M^i$ is the source terms due to the horizontal curvilinear coordinate as

$$F_M^i = -\Gamma^i_{ml} (\rho u^m u^l + G^{ml} p'), \tag{2.19}$$

where $m, l$ take values of $1, 2, 3$. $F_C^i$ is the the Coriolis terms as

$$F_C^i = -G^{im} \epsilon_{jml} 2\Omega^m \rho u^l, \tag{2.20}$$

where $\epsilon_{jkl}$ is the three rank Levi–Civita tensor and $\Omega^m$ are the components of angular velocity vector. $F_{\rm buo}$ is the buoyancy term as

$$F_{\rm buo} = -\rho' \left(\frac{a}{r}\right)^2 g, \tag{2.21}$$

where $r$ is the radial coordinate, $a$ is the planetary radius, and $g$ is the standard gravitational acceleration. In Eq. (2.12), the terms with subscript SGS are associated with a turbulent model; $\boldsymbol{f}_{\mathrm{SGS}}(\boldsymbol{q}, \nabla\boldsymbol{q})$, $\boldsymbol{g}_{\mathrm{SGS}}(\boldsymbol{q}, \nabla\boldsymbol{q})$, and $\boldsymbol{h}_{\mathrm{SGS}}(\boldsymbol{q}, \nabla\boldsymbol{q})$ are the parameterized eddy fluxes while $\boldsymbol{S}_{\mathrm{SGS}}(\boldsymbol{q}, \nabla\boldsymbol{q})$ are the source terms with the curvilinear coordinates. The terms associated with the turbulent model are detailed in Sect. 4.1. On the other hand, $\boldsymbol{S}_{*,\mathrm{phys}}$ represents the terms with physical processes such as cloud, radiation, and surface processes.

To consider the moist thermodynamics, we introduce a potential temperature defined as

$$\theta = T \left(\frac{P_0}{p}\right)^{R^*/C_p^*}, \tag{2.22}$$

where $T$ is the temperature, $P_0$ is a constant pressure. The gas constant $R^*$ and the specific heat at constant volume $C_p^*$ are defined as

$$C_p^* = \sum_{*=d,v,l,s} q_* C_{p,*}, \quad R^* = \sum_{*=d,v,l,s} q_* R_*. \tag{2.23}$$

To close the equation systems in Eq. (2.12), the pressure $p$ is calculated using

$$p = P_0 \left(\frac{R^*}{P_0}\rho\theta\right)^{\frac{C_p^*}{C_v^*}}, \tag{2.24}$$

where $C_v^* = C_p^* - R^*$. In terms of $C_p^*$ and $R^*$, a diabatic heating contribution $H_{\rho\theta}$ included in $\boldsymbol{S}_{\rho\theta,\mathrm{phys}}$ can be written as

$$H_{\rho\theta} = \frac{1}{C_p^*}\left(\frac{P_0}{p}\right)^{\frac{R^*}{C_p^*}} Q, \tag{2.25}$$

where $Q$ is the diabatic heating with the unit $\mathrm{J/(m^3 s)}$.

When the traditional approximation is applied to the governing equations in global model, $\delta_S$ should be set to zero.

# 3 Discretization of dynamics

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## 3.1 Spatial discretization

We perform the spatial discretization for Eq. (2.12) based on a nodal DGM (e.g., Hesthaven and Warburton, 2007). The three-dimensional computational domain $\Omega$ is divided using non-overlapping hexahedral elements. To relate the coordinates $(\xi^1, \xi^2, \xi^3)$ with the local coordinates $\tilde{\boldsymbol{x}} \equiv (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ in a reference element $\Omega_e$, we adopted a linear mapping defined as

$$\tilde{x}^i = 2\frac{\xi^i - \xi^i_e}{h^i_e}, \tag{3.1}$$

where $\xi^i_e$ and $h^i_e$ represent the center position and width of the element, respectively, in the $\xi^i$-direction.

Using the tensor-product of one-dimensional Lagrange polynomials

$$l_{\boldsymbol{m}}(\tilde{\boldsymbol{x}}) = l_{m_1}(\tilde{x}^1) l_{m_2}(\tilde{x}^2) l_{m_3}(\tilde{x}^3), \tag{3.2}$$

a local approximated solution within each element $\Omega_e$ can be represented as

$$\boldsymbol{q}^e|_{\Omega_e}(\tilde{\boldsymbol{x}}, t) = \sum_{m_1=1}^{p+1} \sum_{m_2=1}^{p+1} \sum_{m_3=1}^{p+1} \boldsymbol{Q}^e_{m_1,m_2,m_3}(t) \;\; l_{m_1}(\tilde{x}^1) l_{m_2}(\tilde{x}^2) l_{m_3}(\tilde{x}^3), \tag{3.3}$$

In Eq. (3.3), the coefficients $Q^e_{m_1,m_2,m_3}$ are the unknown degrees of freedom (DOF) and $p$ is the polynomial order. In this study, the Legendre–Gauss–Lobatto (LGL) points were used for interpolation and integration nodes.

#### Semi-discretized equations

By applying the Galerkin approximation to Eq. (2.12), a strong form of the semi-discretized equations can be obtained as

$$\begin{aligned} \frac{D}{Dt} \int_{\Omega_e} \boldsymbol{q}^e(\tilde{\boldsymbol{x}}, t) \; l_{\boldsymbol{m}}(\tilde{\boldsymbol{x}}) \; J^E \; d\tilde{\boldsymbol{x}} = & - \sum_{j=1}^{3} \int_{\Omega_e} \frac{\partial \boldsymbol{F}_j(\boldsymbol{q}^e, \boldsymbol{G})}{\partial \xi^j} l_{\boldsymbol{m}}(\tilde{\boldsymbol{x}}) \; J^E \; d\tilde{\boldsymbol{x}} \\ & - \int_{\partial\Omega_e} \left[ \hat{\boldsymbol{F}}(\boldsymbol{q}^e, \boldsymbol{G}) - \boldsymbol{F}(\boldsymbol{q}^e, \boldsymbol{G}) \right] \cdot \boldsymbol{n} \; l_{\boldsymbol{m}}(\tilde{\boldsymbol{x}}) \; J^{\partial E} \; dS \\ & + \int_{\Omega_e} \left[ \boldsymbol{S}(\boldsymbol{q}^e) + \boldsymbol{S}_{\mathrm{SGS}}(\boldsymbol{q}^e, \boldsymbol{G}) \right] l_{\boldsymbol{m}}(\tilde{\boldsymbol{x}}) \; J^E \; d\tilde{\boldsymbol{x}}, \end{aligned} \tag{3.4}$$

where $(\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3) = (\boldsymbol{f} + \boldsymbol{f}_{\rm SGS}, \boldsymbol{g} + \boldsymbol{g}_{\rm SGS}, \boldsymbol{h} + \boldsymbol{h}_{\rm SGS})$ is the flux vector tensor, $\hat{\boldsymbol{F}}$ is the numerical flux at the element boundary $\partial\Omega_E$, and $\boldsymbol{n}$ is the outward unit vector normal to $\partial\Omega_E$; In the volume and surface integrals, $J^E$ and $J^{\partial E}$ represent the transformation Jacobian with the general curvilinear coordinates and local coordinates within each element. Note that, because of the linear mapping in Eq. (3.1), the associated geometric factors such as $J^E$ and $J^{\partial E}$ have constant values when the volume and surface integrals are calculated. For the turbulent model, we need to evaluate the eddy viscous flux tensor and diffusion flux, which include a few gradient terms with quantities such as $\boldsymbol{\chi} = (u^\xi, u^\eta, u^\zeta, \theta, q_{v,l,s})$, denoted by $\boldsymbol{G} = (\partial\boldsymbol{\chi}/\partial\xi^1, \partial\boldsymbol{\chi}/\partial\xi^2, \partial\boldsymbol{\chi}/\partial\xi^3)$ in Eq. (3.4). The gradient discretization in the $\xi^j$-direction is given by

$$\int_{\Omega_e} \rho\, \boldsymbol{G}_j l_{\boldsymbol{m}}(\tilde{\boldsymbol{x}})\ J^E\ d\tilde{\boldsymbol{x}} = \int_{\Omega_e} \left[ \frac{\partial \rho^e \boldsymbol{\chi}^e}{\partial \xi^j} - \boldsymbol{\chi}^e \left( \frac{\partial \rho}{\partial \xi^j} \right)^e \right] l_{\boldsymbol{m}}(\tilde{\boldsymbol{x}})\ J^E\ d\tilde{\boldsymbol{x}} + \int_{\partial\Omega_e} \left( \widehat{\rho\boldsymbol{\chi}} - \rho^e \boldsymbol{\chi}^e \right) \boldsymbol{n}_{\tilde{x}^j} \cdot \boldsymbol{n}\ l_{\boldsymbol{m}}(\tilde{\boldsymbol{x}})\ J^{\partial E}\ dS, \tag{3.5}$$

where $\boldsymbol{n}_{\tilde{\boldsymbol{x}}^j}$ is the unit vector in the $\tilde{x}^j$-direction and the density gradient is calculated by

$$\int_{\Omega_e} \left( \frac{\partial \rho}{\partial \xi^j} \right)^e l_{\boldsymbol{m}}(\tilde{\boldsymbol{x}})\ J^E\ d\tilde{\boldsymbol{x}} = \int_{\Omega_e} \frac{\partial \rho^e}{\partial \xi^j} l_{\boldsymbol{m}}(\tilde{\boldsymbol{x}})\ J^E\ d\tilde{\boldsymbol{x}} + \int_{\partial\Omega_e} (\hat{\rho} - \rho^e) \boldsymbol{n}_{\tilde{x}^j} \cdot \boldsymbol{n}\ l_{\boldsymbol{m}}(\tilde{\boldsymbol{x}})\ J^{\partial E}\ dS. \tag{3.6}$$

### Numerical flux

For the numerical flux of the inviscid terms, the Rusanov flux (Rusanov, 1961) is used as a simple choice of the approximated Riemann solvers. Its numerical dissipation is provided based on the maximum absolute eigenvalue of the Jacobian matrix at the left and right sides of the element boundary. The Rusanov flux is written as

$$\hat{\boldsymbol{F}}_{\rm invis} = \frac{1}{2} \left\{ \left[ \boldsymbol{F}_{\rm invis}(\boldsymbol{q}^+) + \boldsymbol{F}_{\rm invis}(\boldsymbol{q}^-) \right] \cdot \boldsymbol{n} - \lambda_{\max} \left[ \boldsymbol{q}^+ - \boldsymbol{q}^- \right] \right\}, \tag{3.7}$$

where $\lambda_{\max}$ is the maximum of the absolute value of eigenvalues of the flux Jacobian in the direction $\boldsymbol{n}$, and $\boldsymbol{q}^-$ and $\boldsymbol{q}^+$ represent the interior and exterior values at $\partial\Omega_e$. Previous studies (e.g., Li et al., 2020) formulated the Rusanov flux taken into account the horizontal and vertical coordinate transformations. Based on their works, at the element boundaries in the horizontal directions ($\xi$ and $\eta$), $\lambda_{\max}$ can be represented as

$$\lambda_{\max,\xi} = \left| u^\xi \right| + \sqrt{G_h^{11}} c_s, \quad \lambda_{\max,\eta} = \left| u^\eta \right| + \sqrt{G_h^{22}} c_s, \tag{3.8}$$

where $c_s = [(C_p/C_v)RT]^{1/2}$ is the speed of sound wave. For the vertical direction $\zeta$, $\lambda_{\max}$ can be represented as

$$\lambda_{\max,\zeta} = \left| \widetilde{u^\zeta} \right| + \left[ 1/\sqrt{G_v} + G_v^{13} G_X + G_v^{23} G_Y \right]^{1/2} c_s, \tag{3.9}$$

where $G_X = G_v^{13} G_h^{11} + G_v^{23} G_h^{12}$ and $G_Y = G_v^{13} G_h^{21} + G_v^{23} G_h^{22}$.

We adopt the central flux as the numerical flux of the gradient $\boldsymbol{G}$ and the SGS fluxes $(\boldsymbol{f}_{\rm SGS}, \boldsymbol{g}_{\rm SGS}, \boldsymbol{h}_{\rm SGS})$ with the turbulent model.

### Matrix form of semi-discretized equation

When the same nodes are used for interpolation and integration (i.e., collocation), a matrix form of Eqs. (3.4) and (3.5) can be obtained as

$$\frac{D\boldsymbol{q}^e}{Dt} = -\sum_{j=1}^{3} d_j D_{\tilde{x}^j} \boldsymbol{F}_j(\boldsymbol{q}^e, \boldsymbol{G}) - \sum_{f=1}^{6} s_{\partial\Omega_{e,f}} L_{\partial\Omega_{e,f}} \left[\hat{\boldsymbol{F}}(\boldsymbol{q}^e, \boldsymbol{G}) - \boldsymbol{F}(\boldsymbol{q}^e, \boldsymbol{G})\right] \cdot \boldsymbol{n} + \boldsymbol{S}(\boldsymbol{q}^e) + \boldsymbol{S}_{\mathrm{SGS}}(\boldsymbol{q}^e, \boldsymbol{G}), \tag{3.10}$$

$$\rho\, \boldsymbol{G}_j = d_j D_{\tilde{x}^j} (\rho^e \boldsymbol{\chi}^e) - \boldsymbol{\chi}^e \left(\frac{\partial \rho}{\partial \xi^j}\right)^e + \sum_{f'=1}^{2} s_{\partial\Omega_{e,f'}} L_{\partial\Omega_{e,f'}} \left(\widehat{\rho\boldsymbol{\chi}} - \rho^e \boldsymbol{\chi}^e\right) \boldsymbol{n}_{\tilde{x}^j} \cdot \boldsymbol{n}, \tag{3.11}$$

where $D_{\tilde{x}^j}$ represents the differential matrix for the $\tilde{x}^j$-direction; $L_{\partial\Omega_{e,f}}$ represents the lifting matrix with the surface integral for the $f$-th element surface, and $L_{\partial\Omega_{e,f'}}$ represents the same for the $f'$-th element surface in the gradient operator for the $\tilde{x}^j$-direction. The components of these matrices are given as

$$(D_{\tilde{x}^j})_{\boldsymbol{m},\boldsymbol{m}'} = M^{-1} \int_{\Omega_e} l_{\boldsymbol{m}} \frac{\partial l_{\boldsymbol{m}'}}{\partial \tilde{x}_j}\, d\tilde{\boldsymbol{x}}, \quad (L_{\partial\Omega_e,j})_{\boldsymbol{m},\boldsymbol{m}'} = M^{-1} \int_{\partial\Omega_{e,j}} l_{\boldsymbol{m}} l_{\boldsymbol{m}'}^{\partial\Omega_{e,j}} dS, \tag{3.12}$$

where $M$ denotes the mass matrix and is given by

$$M_{\boldsymbol{m},\boldsymbol{m}'} = \int_{\Omega_e} l_{\boldsymbol{m}} l_{\boldsymbol{m}'}\, d\tilde{\boldsymbol{x}}. \tag{3.13}$$

The density gradient term is calculated as

$$\left(\frac{\partial \rho}{\partial \xi^j}\right)^e = d_j D_{\tilde{x}^j} \rho^e - \sum_{f'=1}^{2} s_{\partial\Omega_{e,f'}} L_{\partial\Omega_{e,f'}} \left(\widehat{\rho} - \rho^e\right) \boldsymbol{n}_{\tilde{x}^j} \cdot \boldsymbol{n}. \tag{3.14}$$

Note that, in Eqs. (3.10), (3.11), and (3.14), $d_j = \partial\tilde{x}^j/\partial\xi^j$ and $s_{\partial\Omega_{e,f'}} = J_{\partial\Omega_{e,f'}}/J^E$ are constant values in the volume and surface integrals, respectively.

## 3.2 Temporal discretization

The semi-discretized equations in Eq. (3.4) can be represented as the following ordinary differential equation (ODE) system

$$\frac{d\boldsymbol{q}}{dt} = \mathcal{S}(\boldsymbol{q}, \nabla\boldsymbol{q}) + \mathcal{F}(\boldsymbol{q}, \nabla\boldsymbol{q}), \tag{3.15}$$

where $\mathcal{S}(\boldsymbol{q}, \nabla\boldsymbol{q})$ and $\mathcal{F}(\boldsymbol{q}, \nabla\boldsymbol{q})$ represent the tendencies with slow and fast contributions, respectively. This study adopted Runge–Kutta (RK) schemes to solve the ODE system from $t = n\Delta t$ to $t = (n+1)\Delta t$, where $\Delta t$ is the time step and $n$ is a natural number. In this subsection, we describe two approaches for temporal discretization, namely, horizontal explicit and vertical implicit (HEVI) and horizontal explicit and vertical explicit (HEVE) approaches.

**HEVI approach**

If the aspect ratio of horizontal grid spacing to its vertical counterpart is large, it is impractical to use fully explicit temporal schemes because the vertically propagating sound waves severely restrict the timestep. A strategy to avoid computational cost in such case is the HEVI approach. The terms corresponding to vertical dynamics with a fast time-scale are evaluated using an implicit temporal scheme, while the remaining terms are evaluated using an explicit temporal scheme. This procedure is regarded as a framework of implicit-explicit (IMEX) time integration scheme (Bao et al., 2015; Gardner et al., 2018). General formulation of IMEX RK scheme (e.g., Ascher et al., 1997) with $\nu$ stages can be represented as

$$\begin{aligned} \boldsymbol{q}^{(s)} &= \boldsymbol{q}^n + \Delta t \sum_{s'=1}^{s-1} a_{ss'} \mathcal{S}(t + c_{s'}\Delta t, \boldsymbol{q}^{(s')}) + \Delta t \sum_{s'=1}^{s} \tilde{a}_{ss'} \mathcal{F}(t + \tilde{c}_{s'}\Delta t, \boldsymbol{q}^{(s')}) \quad \text{for} \quad s = 1, \ldots, \nu \\ \boldsymbol{q}^{n+1} &= \boldsymbol{q}^n + \Delta t \sum_{s=1}^{\nu} b_s \mathcal{S}(t + c_s\Delta t, \boldsymbol{q}^{(s)}) + \Delta t \sum_{s=1}^{\nu} \tilde{b}_s \mathcal{F}(t + \tilde{c}_s\Delta t, \boldsymbol{q}^{(s)}), \end{aligned} \tag{3.16}$$

where $a_{ss'}$, $b_s$, and $c_s$ define the explicit temporal integrator, while $\tilde{a}_{ss'}$, $\tilde{b}_s$, and $\tilde{c}_{s'}$ define the implicit temporal integrator; $c_s = \sum_{s'=1}^{s-1} a_{ss'}$ and $\tilde{c}_s = \sum_{s'=1}^{s-1} \tilde{a}_{ss'}$ represents time when slow and fast terms are evaluated, respectively. These coefficients are compactly represented using "double Butcher tableaux", as shown in Table A.1. Note that, in the table of the explicit part, $\mathscr{A} = \{a_{ss'}\}$ with $a_{ss'} = 0$ for $s' \geq s$. On the other hand, for the implicit part, $\tilde{\mathscr{A}} = \{\tilde{a}_{ss'}\}$ with $\tilde{a}_{ss'} = 0$ for $s' > s$ in the case of the diagonally implicit RK scheme.

The terms associated with vertical mass flux, vertical pressure gradient, vertical flux of potential temperature, and buoyancy in Eq. (2.12) were treated as fast terms, whereas the other terms were treated as slow terms.

In SCALE-DG, several implicit and explicit (IMEX) RK schemes in Table 3.1 can be available. To minimize contaminating the spatial accuracy of high-order DGM by temporal errors present in low-order HEVI scheme, a third-order scheme proposed by Kennedy and Carpenter (2003) was adopted in our previous studies (Kawai and Tomita, 2025); it includes four explicit and three implicit evaluations. The corresponding double Butcher tableaux are given in Table A.1.

Table 3.1: Implicit and Explicit (IMEX) Runge–Kutta schemes available in HEVI temporal integration. For the number of RK stages, (I,E) represents implicit and explicit parts, respectively.

| Abbrev. | Order | Num. of stages (I,E) | Note | Reference |
|---|---|---|---|---|
| IMEX_ARK232 | 2 | (2,3) | | Giraldo et al. (2013) |
| IMEX_ARK324 | 3 | (3,4) | | Kennedy and Carpenter (2003) |

Table 3.2: Explicit Runge–Kutta schemes available in HEVE temporal integration

| Abbrev. | Order | Num. of stages | Note | Reference |
|---|---|---|---|---|
| ERK_Euler | 1 | 1 | for debug | |
| ERK_SSP_2s2o | 2 | 2 | SSP | Shu and Osher (1988) |
| ERK_SSP_3s3o | 3 | 3 | SSP | Shu and Osher (1988) |
| ERK_SSP_4s3o | 3 | 4 | SSP | |
| ERK_SSP_5s3o_2N2* | 3 | 5 | SSP | Higueras and Roldán (2019) |
| ERK_RK4 | 4 | 4 | classical RK4 | |
| ERK_SSP_10s4o_2N | 4 | 10 | SSP | Ketcheson (2008) |

In the implicit part of each stage, the corresponding nonlinear equation system is solved using Newton's method. In each iteration, the linearized equation system is solved. Obtaining accurate solutions of the nonlinear equation system generally requires numerous iterations. However, this study performed a single iteration in Newton's method (i.e., Rosenbrock approach), significantly reducing the computational cost. Similar approach has been used in previous studies (Ullrich and Jablonowski, 2012a). In the case of the collocation approach, because the horizontal dependency between all nodes within an element vanishes, the vertical implicit evaluation can be parallelly performed at each horizontal node.

For the case of HEVI, the volume and surface integrations in Eqs. (3.12) and (3.13) were evaluated using inexact integration with the LGL nodes. Consequently, $\boldsymbol{M}$ and $\boldsymbol{L}_{\partial\Omega_{e,3}}$ became diagonal matrices, which further simplified the matrix structure associated with the vertical spatial operator.

**HEVE approach**

When we consider a horizontal grid spacing with O(10 m) such as in LES, the ratio of horizontal to vertical grid spacing approaches unity. The advantages of HEVI approach decrease. Thus, it is suitable to adopt a fully explicit temporal approach, referred to as HEVE approach. In such cases, RK schemes with a strong stability preserving (SSP) property (Gottlieb et al., 2001) are often used in combination with DGM. In SCALE-DG, several RK schemes in Table 3.2 can be available. The corresponding Butcher table and coefficients with Shu-Osher form are shown in Appendix A.1. In our previous studies (Kawai and Tomita, 2023, 2025), we adopted a ten-stage RK scheme with the fourth-order accuracy proposed by Ketcheson (2008).

When using the HEVE approach, entries of the matrices in Eqs. (3.12) and (3.13) were directly calculated following Sect. 3.2 in Hesthaven and Warburton (2007).

## 3.3 Additional stabilization

### 3.3.1 Modal filtering

For high-order DGM, numerical instability is likely to occur in advection-dominated flows because the numerical dissipations with the upwind numerical fluxes weaken. Furthermore, we adopted a collocation approach due to its computational efficiency. One drawback is that the aliasing errors with evaluations of the nonlinear terms can drive numerical instability. To suppress this numerical instability, a modal filter was used as an additional stabilization mechanism. The filter matrix for the three-dimensional problem can be obtained as

$$\mathscr{F} = V^{\mathrm{3D}} C^{\mathrm{3D}} V^{\mathrm{3D}}, \tag{3.17}$$

where $V^{\mathrm{3D}}$ represents the Vandermode matrix associated with the LGL interpolation nodes (in Eq. (3.3)) and $C^{\mathrm{3D}}$ represents the diagonal cutoff matrix. The entries of $C^{\mathrm{3D}}$ are defined as

$$C^{\mathrm{3D}}_{(m_1,m_2,m_3),(m_1',m_2',m_3')} = \delta_{m_1,m_1'} \sigma^h_{m_1} \; \delta_{m_2,m_2'} \sigma^h_{m_2} \; \delta_{m_3,m_3'} \sigma^v_{m_3}, \tag{3.18}$$

where $\sigma^h_i$ and $\sigma^v_i$ represent the decay coefficient for the one-dimensional horizontal and vertical modes $i$, respectively. Based on Hesthaven and Warburton (2007), a typical choice of the coefficient for mode $i$ is provided with an exponential function as

$$\sigma_i = \begin{cases} 1 & \text{if } \; 0 \le i \le p_c \\ \exp\left[ -\alpha_m \left( \dfrac{i - p_c}{p - p_c} \right)^{p_m} \right] & \text{if } \; p_c \le i \le p, \end{cases} \tag{3.19}$$

where $p_c$, $p_m$, and $\alpha_m$ represent the cutoff parameter, the order of the filter, and the non-dimensional decay strength, respectively. In this study, $p_c$ was considered 0. We applied the filter $\mathscr{F}$ to the solution vector $\boldsymbol{q}$ (in Eq. (2.13)) at the final stage of the RK scheme with a timestep $\Delta t$. Then, the decay time scale for the highest mode can be regarded as approximately equal to $\Delta t / \alpha_m$. We set the order $p_m$, and decay coefficient $\alpha_m$ such that the strength of filter should ensure numerical stability while being as weak as possible.

### 3.3.2 Buoyancy term

The balance between the pressure gradient and buoyancy terms should be carefully treated in the discrete momentum equation (e.g., Blaise et al., 2016; Orgis et al., 2017). In the above formulation, because a different discretization space is used between the terms, a numerical imbalance is possible and may cause spurious oscillations, which can destabilize the simulations. To avoid this incompatibility, the vertical polynomial order for the density in the buoyancy term was reduced by one following Blaise et al. (2016).

# 4 Physical parameterization

## 4.1 Turbulence

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### 4.1.1 Smagorinsky-type model

As a turbulent model, this subsection describes a Smagorinsky–Lilly type model (Smagorinsky, 1963; Lilly, 1962) that considered the stratification effect (Brown et al., 1994). As a spatial filter, the Favre-filtering (Favre, 1983) was used. We do not explicitly denote the symbol representing the spatial filter because the filtering approach is essentially the same as that explained in Appendix A of Kawai and Tomita (2023). The difficulties in the derivation of viscous and diffusion terms are caused by the gradient of vector quantities and the spatial divergence with the non-orthogonal basis because the manipulations grow increasingly complex. However, previous studies that utilized tensor analysis help us provide a systematic derivation (e.g., Ullrich, 2014; Rančić et al., 2017). In the absence of a vertical coordinate transformation, the parameterized fluxes with the turbulent model can be represented in the general curvilinear coordinates as

$$\boldsymbol{f}_{\mathrm{SGS}}(\boldsymbol{q},\nabla\boldsymbol{q}) = \begin{pmatrix} 0 \\ -\sqrt{G}\rho\tau^{11} \\ -\sqrt{G}\rho\tau^{12} \\ -\sqrt{G}\rho\tau^{13} \\ -\sqrt{G}\rho\tau^{1}_{*} \end{pmatrix}, \quad \boldsymbol{g}_{\mathrm{SGS}}(\boldsymbol{q},\nabla\boldsymbol{q}) = \begin{pmatrix} 0 \\ -\sqrt{G}\rho\tau^{21} \\ -\sqrt{G}\rho\tau^{22} \\ -\sqrt{G}\rho\tau^{23} \\ -\sqrt{G}\rho\tau^{2}_{*} \end{pmatrix}, \tag{4.1}$$

$$\boldsymbol{h}_{\mathrm{SGS}}(\boldsymbol{q},\nabla\boldsymbol{q}) = \begin{pmatrix} 0 \\ -\sqrt{G}\rho\tau^{31} \\ -\sqrt{G}\rho\tau^{32} \\ -\sqrt{G}\rho\tau^{33} \\ -\sqrt{G}\rho\tau^{3}_{*} \end{pmatrix}, \tag{4.2}$$

and the source term can be given by

$$\boldsymbol{S}_{\mathrm{SGS}}(\boldsymbol{q},\nabla\boldsymbol{q}) = \begin{pmatrix} 0 \\ -\sqrt{G}\Gamma^{1}_{ml}\rho\tau^{ml} \\ -\sqrt{G}\Gamma^{2}_{ml}\rho\tau^{ml} \\ -\sqrt{G}\Gamma^{3}_{ml}\rho\tau^{ml} \\ 0 \end{pmatrix}. \tag{4.3}$$

In the equations, $\tau^{ij}$ is the contravariant components of parameterized eddy viscous flux tensor ($i = 1, 2, 3$ and $j = 1, 2, 3$) and can be written as

$$\tau^{ij} = -2\nu_{\mathrm{SGS}} \left( S^{ij} - \frac{G^{ij}}{3} D \right) - \frac{2}{3} G^{ij} K_{\mathrm{SGS}}, \tag{4.4}$$

where $S^{ij}$ is the strain velocity tensor, $\nu_{\mathrm{SGS}}$ is the eddy viscosity, $D$ is the divergence of the three-dimensional velocity, and $K_{\mathrm{SGS}}$ is the SGS kinetic energy. The strain velocity tensor is represented as

$$S^{ij} = \frac{1}{2} \left( G^{im} \frac{\partial u^j_{,m}}{\partial \xi^m} + G^{jm} \frac{\partial u^i_{,m}}{\partial \xi^m} \right), \tag{4.5}$$

using the covariant derivative of the contravariant velocity component

$$u^i_{,j} = \frac{\partial u^i}{\partial \xi^j} + u^m \Gamma^i_{jm}. \tag{4.6}$$

The eddy viscosity is written as

$$\nu_{\mathrm{SGS}} = C_s \Delta_{\mathrm{SGS}} |S|, \tag{4.7}$$

where $C_s$, $\Delta_{\mathrm{SGS}}$, and $|S|$ represent the Smagorinsky constant, the filter length, and the norm of strain tensor defined as $\sqrt{2 G_{im} G_{jn} S^{ij} S^{mn}}$, respectively. The parameterized eddy diffusive flux can be written as

$$\tau^i_* = -\nu^*_{\mathrm{SGS}} G^{ij} \frac{\partial \theta}{\partial \xi^j}, \tag{4.8}$$

where $\nu^*_{\mathrm{SGS}}$ is the eddy diffusion coefficient. For further details of the turbulent model, refer to Sect. 2.2 of Nishizawa et al. (2015).

# A The detail numerics

## A.1 Runge–Kutta scheme

This section provides Butcher tables and coefficients with Shu–Osher form of Runge–Kutta schemes.

Table A.1: Double Butcher table for a third-order IMEX RK scheme proposed by Kennedy and Carpenter (2003).

| $c_s$ | $a_{ss'}$ | | | |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $\frac{1767732205903}{2027836641118}$ | $\frac{1767732205903}{2027836641118}$ | $0$ | $0$ | $0$ |
| $\frac{3}{5}$ | $\frac{5535828885825}{10492691773637}$ | $\frac{788022342437}{10882634858940}$ | $0$ | $0$ |
| $1$ | $\frac{6485989280629}{16251701735622}$ | $-\frac{4246266847089}{9704473918619}$ | $-\frac{10755448449292}{10357097424841}$ | $0$ |
| $b_s$ | $\frac{1471266399579}{7840856788654}$ | $-\frac{4482444167858}{7529755066697}$ | $\frac{11266239266428}{11593286722821}$ | $\frac{1767732205903}{4055673282236}$ |
| $\tilde{c}_s$ | $\tilde{a}_{ss'}$ | | | |
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $\frac{1767732205903}{2027836641118}$ | $\frac{1767732205903}{4055673282236}$ | $\frac{1767732205903}{4055673282236}$ | $0$ | $0$ |
| $\frac{3}{5}$ | $\frac{2746238789719}{10658868560708}$ | $-\frac{640167445237}{6845629431997}$ | $\frac{1767732205903}{4055673282236}$ | $0$ |
| $1$ | $\frac{1471266399579}{7840856788654}$ | $-\frac{4482444167858}{7529755066697}$ | $\frac{1767732205903}{11593286722821}$ | $\frac{1767732205903}{4055673282236}$ |
| $\tilde{b}_s$ | $\frac{1471266399579}{7840856788654}$ | $-\frac{4482444167858}{7529755066697}$ | $\frac{11266239266428}{11593286722821}$ | $\frac{1767732205903}{4055673282236}$ |

Table A.2: Butcher table and Shu-Osher form for a third-order fully explicit SSP RK scheme with three stages

| $c_s$ | $a_{ss'}$ | | |
|---|---|---|---|
| $0$ | | | |
| $1$ | $1$ | | |
| $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | |
| $b_s$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{2}{3}$ |

| $\alpha_{ss'}$ | | |
|---|---|---|
| $1$ | | |
| $\frac{3}{4}$ | $\frac{1}{4}$ | |
| $\frac{1}{3}$ | $0$ | $\frac{2}{3}$ |

| $\beta_{ss'}$ | | |
|---|---|---|
| $1$ | | |
| $0$ | $\frac{1}{4}$ | |
| $0$ | $0$ | $\frac{2}{3}$ |

Table A.3: Butcher table and Shu-Osher form for a third-order fully explicit SSP RK scheme with four stages

| $c_s$ | $a_{ss'}$ | | | |
|---|---|---|---|---|
| $0$ | | | | |
| $\frac{1}{2}$ | $\frac{1}{2}$ | | | |
| $1$ | $\frac{1}{2}$ | $\frac{1}{2}$ | | |
| $\frac{1}{2}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | |
| $b_s$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{2}$ |

| $\alpha_{ss'}$ | | | |
|---|---|---|---|
| $1$ | | | |
| $0$ | $1$ | | |
| $\frac{2}{3}$ | $0$ | $\frac{1}{3}$ | |
| $0$ | $0$ | $0$ | $1$ |

| $\beta_{ss'}$ | | | |
|---|---|---|---|
| $\frac{1}{2}$ | | | |
| $0$ | $\frac{1}{2}$ | | |
| $0$ | $0$ | $\frac{1}{6}$ | |
| $0$ | $0$ | $0$ | $\frac{1}{2}$ |

Table A.4: Shu-Osher form for a third-order and five-stage fully explicit SSP RK scheme with $2N_2^*$ storage

| $\alpha_{ss'}$ | | | | |
|---|---|---|---|---|
| 1 | | | | |
| 0 | 1 | | | |
| 0.682342861037239 | 0 | 0.317657138962761 | | |
| 0 | 0 | 0 | 1 | |
| 0.045230974482400 | 0 | 0 | 0 | 0.954769025517600 |

| $\beta_{ss'}$ | | | | |
|---|---|---|---|---|
| 0.465388589249323 | | | | |
| 0 | 0.465388589249323 | | | |
| 0 | 0 | 0.124745797313998 | | |
| 0 | 0 | 0 | 0.465388589249323 | |
| 0 | 0 | 0 | 0 | 0.154263303748666 |

Table A.5: Butcher table for a fourth-order fully explicit RK scheme with ten stages porposed by Ketcheson (2008).

| $c_s$ | $a_{ss'}$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | | | |
| $\frac{1}{6}$ | $\frac{1}{6}$ | | | | | | | | | |
| $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | | | | | | | | |
| $\frac{1}{2}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | | | | | | | |
| $\frac{2}{3}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | | | | | | |
| $\frac{1}{3}$ | $\frac{1}{15}$ | $\frac{1}{15}$ | $\frac{1}{15}$ | $\frac{1}{15}$ | $\frac{1}{15}$ | | | | | |
| $\frac{1}{2}$ | $\frac{1}{15}$ | $\frac{1}{15}$ | $\frac{1}{15}$ | $\frac{1}{15}$ | $\frac{1}{15}$ | $\frac{1}{6}$ | | | | |
| $\frac{2}{3}$ | $\frac{1}{15}$ | $\frac{1}{15}$ | $\frac{1}{15}$ | $\frac{1}{15}$ | $\frac{1}{15}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | | | |
| $\frac{5}{6}$ | $\frac{1}{15}$ | $\frac{1}{15}$ | $\frac{1}{15}$ | $\frac{1}{15}$ | $\frac{1}{15}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | | |
| 1 | $\frac{1}{15}$ | $\frac{1}{15}$ | $\frac{1}{15}$ | $\frac{1}{15}$ | $\frac{1}{15}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | |
| $b_s$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ |

| $\alpha_{ss'}$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | |
| 0 | 1 | | | | | | | | |
| 0 | 0 | 1 | | | | | | | |
| 0 | 0 | 0 | 1 | | | | | | |
| $\frac{3}{5}$ | 0 | 0 | 0 | $\frac{2}{5}$ | | | | | |
| 0 | 0 | 0 | 0 | 0 | 1 | | | | |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | | | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

| $\beta_{ss'}$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\frac{1}{6}$ | | | | | | | | | |
| 0 | $\frac{1}{6}$ | | | | | | | | |
| 0 | 0 | $\frac{1}{6}$ | | | | | | | |
| 0 | 0 | 0 | $\frac{1}{6}$ | | | | | | |
| 0 | 0 | 0 | 0 | $\frac{1}{15}$ | | | | | |
| 0 | 0 | 0 | 0 | 0 | $\frac{1}{6}$ | | | | |
| 0 | 0 | 0 | 0 | 0 | 0 | $\frac{1}{6}$ | | | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{1}{6}$ | | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{1}{6}$ | |
| 0 | 0 | 0 | 0 | $\frac{3}{50}$ | 0 | 0 | 0 | 0 | $\frac{1}{10}$ |

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